# TWO-DIMENSIONAL WAVE PROCESSES IN A DISPERSIVE MEDIUM

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One-dimensional problems of the propagation and evolution of waves in dispersive media have been studied in detail in [1-3]. Similar problems in the two-dimensional formulation have been studied to a considerably lesser extent [4, 5], due to the considerable difficulties involved. In [4] the stability of two-dimensional solutions was studied analytically for the case when there was amplitude modulation. The stationary two-dimensional problem of the flow round a thin body in a dispersive medium when the dispersion and nonlinear effects were small was solved analytically in [5].

To consider two-dimensional wave processes in a dispersive medium without assuming the nonlinear and dispersive effect to be small we chose the simplest model, namely, the flow of a supersonic collision-free nonisothermal plasma around a conducting infinite cylinder. It is well known that in such a plasma iono-sonic waves can propagate, the velocity of which depends on the wavelength (in other words, a plasma with $T_e \gg T_i$ is a dispersive medium for iono-sonic waves).

Ion-acoustic waves in a nonisothermal plasma without a magnetic field for amplitudes less than the critical values [1] can be described by the following hydrodynamic equations:

$$\frac{\partial n_i}{\partial t} + \operatorname{div}(n_i \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{e}{m_i}\nabla\varphi - \frac{1}{n_i m_i}\nabla p_i + \frac{\nu}{n_i m_i}\Delta\mathbf{u},$$

$$\Delta\varphi = 4\pi e(n_e - n_i), \quad p_i = n_i T_i, \quad n_e = n_0 \exp(e\varphi/T_e),$$

where $\mathbf{u}_i$, $m_i$, $n_i$, $T_i$ are the velocity, mass, density, and temperature of the ions, $n_e$ and $T_e$ are the density temperature of the electrons, $\varphi$ is the electric potential, $T_e \gg T_i$, and $\nu$ is the viscosity. When there is no viscosity the dispersion oscillations will eventually fill the more and more expanding region in front of the body. The introduction of viscosity enables one to limit this region. We will solve the problem in polar coordinates (r, $\theta$) in the region $R_0 \leqslant r \leqslant R_1$, $\theta_0 \leqslant \theta \leqslant \pi$. As the initial data over the whole region, in addition to the surface of the cylinder, we will specify the parameters of the uniform leading flow

$$n_i(\mathbf{r}, 0) = n_0, \ \mathbf{u}(\mathbf{r}, 0) = \mathbf{u}_0 = \text{const}, \ \varphi(\mathbf{r}, 0) = 0, \tag{2}$$

where $n_0$, $\mathbf{u}_0$ are the values of the density and velocity of the plasma flow at infinity, which is assumed to be supersonic, i.e., $u_0 > c_s = (T_e/m_i)^{1/2}$. On the surface of the cylinder we will specify the condition

$$\varphi(R_0, t) = \varphi_0 = \text{const}. \tag{3}$$

The external boundary of the region $r = R_1$ will be assumed to be fairly far from the body, so that

$$\varphi(R_1, t) = 0. \tag{4}$$

The boundary conditions for the hydrodynamic functions are as follows:

$$\mathbf{u}(R_0, t) = 0, \ \mathbf{u}(R_1, t) = \mathbf{u}_0, \ n_i(R_1, t) = n_0. \tag{5}$$

The values of the potential $\varphi_0$ on the body, and also the electron and ion temperatures $T_e$ and $T_i$ will be assumed to be certain constants. Note that in the experiment the potential on the body can be varied, while the constancy of the temperatures in problems of ion-acoustic oscillations is a normal assumption. Hence, Eqs. (1) with conditions (2) and (3)-(5) formulate the problem mathematically.

In this problem there are three spatial scales: the dimensions of the body, the electron Debye radius $r_{De} = (T_e/4\pi n_0 e^2)^{1/2}$ (characterizing the scale of the dispersion oscillations), and the ion Debye radius $r_{Di} = (T_i/4\pi n_0 e^2)^{1/2}$ (characterizing the effect of the charge of the body). It is convenient to choose the radius of the cylinder $R_0$ around which the flow occurs as the fundamental spatial scale. We will choose as the characteristic values of the velocity, density, and potential, the velocity of ion sound $c_s = (T_e/m_i)^{1/2}$, $n_0$, and $T_e/e$ respectively. Then, the system of equations (1) in dimensionless variables and in polar coordinates (r, $\theta$) can be written in the form

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$$\frac{\partial n}{\partial t}+u\frac{\partial n}{\partial r}+\frac{v}{r}\frac{\partial n}{\partial \theta}+n\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{nu}{r}=0, \quad (6)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+\frac{\partial \varphi}{\partial r}+\frac{T}{n}\frac{\partial n}{\partial r}=\frac{\nu}{n}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right),$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+\frac{1}{r}\frac{\partial \varphi}{\partial \theta}+\frac{T}{nr}\frac{\partial n}{\partial \theta}=\frac{\nu}{n}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}\right),$$

$$\beta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right]=\exp(\varphi)-n,$$

where u is the radial component of the velocity, v is the azimuthal component of the velocity, $\beta=(r_{De}/R_0)^2$, and $T = T_i/T_e$. The system of equations (6) was solved numerically by the method of fractional steps. All terms of the transfer were approximated taking into account the direction of the velocity; the difference scheme was monotonic and conditionally stable. The property of monotonicity in this case is very important since if this is not satisfied "numerical" dispersion can distort the physical dispersion. Poisson's equation for the potential was solved by the quasi-linearization method with subsequent iterations using the upper-relaxation algorithm.

We will consider the results of the calculations in the whole region $R_0 \leqslant r \leqslant 3R_0$, $0 \leqslant \theta \leqslant \pi(\theta_0 = 0)$ when there is no viscosity ($\nu = 0$). The velocity of the leading flow is chosen to be $u_0 = 1, 2$, and the dispersion parameter $\beta = 0.09$, the ratio of the ion and electron temperatures T = 0.1, and the potential of the charged body $\varphi_0 = 0.5$. The choice of the complete region $0 \leqslant \theta \leqslant \pi$ enables us to investigate not only the structure of the shockwave in front of the body, but also the rarefied trail behind it. According to the calculations, a condensation region with maximum density $n_{max} \approx 18.5\ n_0$, is formed in front of the body, while behind the body there is an extremely rarefied trail with $n_{min} = 10^{-4}\ n_0$. Figure 1 shows the angular dependence of the perturbation of the ion density $\delta n_i=(n_i-n_0)/n_0$ at a distance $r = 2R_0$ from the cylinder axis at the instant of time $t = R_0/c_s$ (the circle corresponds to the unperturbed density; outside the circle we have plotted $\delta n_i > 0$ as a function of the angle $\theta$, and inside the circle $\delta n_i < 0$). The shock wave is transferred to the body by the incoming plasma flow. The increase in density when $\theta \approx 2$ corresponds to the Mach cone. Behind the body a rarefied trail is formed in which the ions are focused around the axis of symmetry $\theta = 0$. Qualitatively similar results were obtained in calculations using kinetic theory [6]. Figure 2 shows isolines of the ion density $n_i(r, t)$; it can be seen that in front of the cylinder there is a shock wave with an oscillatory structure.

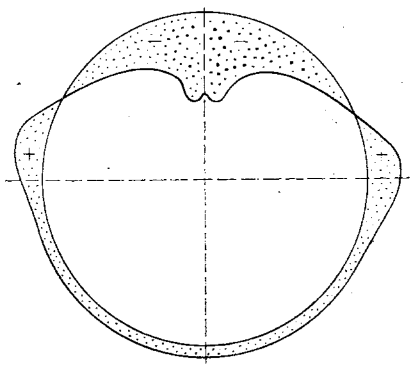


Fig. 1

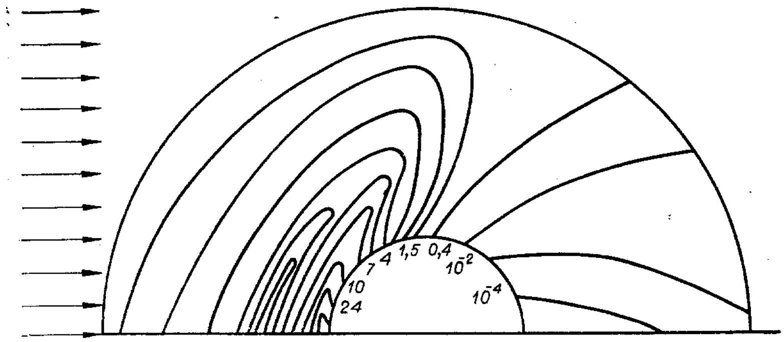


Fig. 2

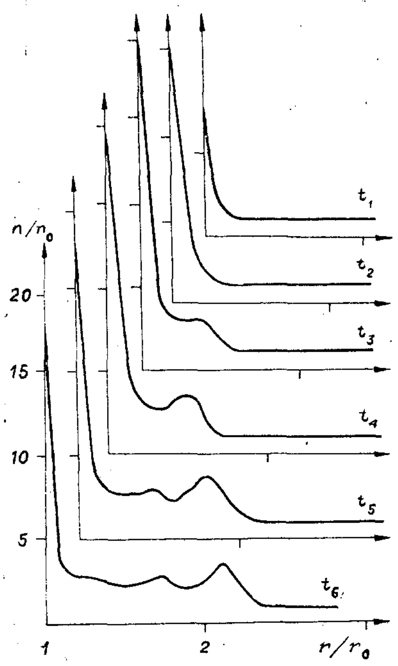


Fig. 3

We will now consider the formation of the shock wave using calculations carried out in the "curtailed" region $R_0 \leqslant r \leqslant 3R_0$, $1 \leqslant \theta \leqslant \pi$; $(\theta_0 = 1)$ when there is slight viscosity $\nu = 10^{-2}$ and with the following parameters: $\beta=0.01$, $\varphi_0=0.5\ T_e/e$, $T_i/T_e=0.1, u_0=1.2\,c_s$. Figure 3 shows the ion density profile on the braking line $\theta = \pi$ at subsequent instants of time. Initially there is an increase in the ion density on the cylinder surface up to a value of $\approx 20\ n_0$ (the instants of time $t_1=0.1\,R_0/c_s$, and $t_2=0.3\,R_0/c_s$). Then, when sufficient density has been stored on the surface of the body, ion pressure begins to have an effect, and "repels" the ions from the surface (instant of time $t_3 = 0.4\,R_0/c_s$), and it begins to leave the cylinder upwards along the flow of the compression wave. Since the medium is dispersive, these perturbations gradually acquire an oscillatory form (instants of time $t_4 = 0.6\,R_0/c_s$, $t_5 = 0.9\,R_0/c_s$ and $t_6 = 1.15\,R_0/c_s$). When the dispersion parameter $\beta$ is reduced considerable oscillations occur, the spatial scale of which decreases in proportion to $\beta^{1/2}$.

When the potential of the cylinder $\varphi_0$ is reduced there is an increase in the ion density on the surface of the body, which occurs due to weakening of the force on the part of the electric field.

When the nonisothermal characteristic of the plasma $T_i/T_e$ is reduced from 0.1 to 0.01 the ion density on the cylinder surface increases from 32 $n_0$ to 224 $n_0$ for $\beta = 10^{-3}$, $\varphi_0 = 0.5\ T_e/e$, and $\nu = 10^{-2}$ at the instant of time $t = R_0/c_s$. This is due to the fact that the "elasticity" of the medium with respect to compression is determined by the temperature ratio $T_i/T_e$, and when this ratio is reduced, the "elasticity" is reduced. In order for ion pressure to play a part, greater condensation on the surface of the body is required. If the velocity of the leading flow $u_0$ is increased, there will also be an increase in the density of the cylinder surface.

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